Rutgers University: Algebra Written Qualifying Exam

January 2008: Day 1 Problem 1 Solution

Exercise. Prove that there are no simple groups of order 12.

Solution.

Let G be a group of order 12. We want to show that there is a normal subgroup of G that is ***not*** $\{e\}$ or G.

$$12 = 2^2 \cdot 3$$

By the third Sylow theorem,

$$\begin{array}{lllll} n_3 \equiv 1 \mod 3 & \text{and} & n_3 \mid 4 & \Longrightarrow & n_3 = 1 \text{ or } 4 \\ n_2 \equiv 1 \mod 2 & \text{and} & n_2 \mid 3 & \Longrightarrow & n_2 = 1 \text{ or } 3 \end{array}$$

If there is only $n_3 = 1$ Sylow 3-subgroups, then the Sylow 3 subgroup is normal in G by the second Sylow theorem.

$\Longrightarrow$ G is not simple

Otherwise, $n_3 = 4$, and G has $4(3-1) = 8$ elements of order 3.

There are $12 - 8 = 4$ remaining elements in G

This is a single Sylow 2-subgroup

$\Longrightarrow$ there must be only one Sylow 2-subgroup

$\Longrightarrow$ the Sylow 2 subgroup is normal in G by the second Sylow theorem

$\Longrightarrow$ G is not simple

Thus, there are no simple groups of order 12.